International Journal of Scientific & Engineering Research, Volume 11, Issue 3, March-2020
ISSN 2229-5518

Simple Approach to Analysis of Thin Laminated Rectangular Plate with Exact Displacement Functions

Ibearugbulem, O.M[1]., Ezeh, J.C[1]., Anya, U. C[1] Okoroafor, S. U[1].,
[1]Department of Civil Engineering, Federal University of Technology, Owerri.

Abstract— This research aimed at analysis of thin laminated rectangular plate with exact displacement functions. The displacement filed and kinematics of the laminas were considered and the total potential energy of the system were applied in generating the total potential energy of the thin laminated plate. The equation for analysis of pure bending, free vibration (natural frequency), and buckling of thin laminated plate were obtained. Numerical examples were carried by considering four layers laminated thin plate with different orientation and different aspect ratio, flexural rigidity and poison ratio for SSSS Plate. The results were compared with the work of other researchers that assumed shape function and the percentage different is less than 0.06% for natural frequency, less than 0.1% for buckling load. With reference to the results obtained in this research, engineers designing thin rectangular laminated plate structures with the aim of obtaining exact solution can do that with ease.

Index Terms— Thin rectangular laminas, buckling, natural frequency, pure bending, total potential energy.

1 Introduction

The analysis of rectangular laminated plates has been characterized with the use of assumed displacement functions [1]; [2]; [3]; [4]. The earlier scholars usually assumed displacement functions that satisfy the stipulated boundary conditions. Little efforts have been made to integrate the governing equation (otherwise called Euler-Bernoulli equilibrium equation of forces) to obtain the displacement functions. It is only when the governing equations are integrated and displacement satisfying it are obtained can one boldly accept to have obtained an exact displacement function for any arbitrary problem in question. This circumventive (evading the integration of the governing equation and instead assuming a displacement function) approach to analysis led the complexities encountered during the analysis. This makes the analysis look cumbersome. Most times Navier's approach or Levy's approach was adopted. This analysis shall be appealing when it looks straight forward and simply. However, this simplicity seemed elusive from most works [5]; [6]; [7]; [8]; [9] reviewed in the course of authoring this paper.

Most works on laminated composite plate are on thick plate ([10], [11]; [12]; [13]; [14]; [15]. The present study shall concentrate on thin laminated plate. Though thin laminated plate attracted little attention from the works reviewed so far ([16]; [17]; [18]; [3]; [19]). The reason for low patronage of thin limited plate for structural use may be attributed to the fact that most structural laminated composite plates are thick.

This reason is not justifiable as many laminated structural composite plates produced in recent times can be adjudged thin since their span-to-depth ratio are somewhat more than 70. It became pertinent to invest energy and resources in evolving simpler ways of analyzing thin rectangular laminated composite structural plates.

2.0 Displacement Field and Kinematics of a Lamina of Thin Laminated Plate

The displacement field include deflection and the two in-plane displacements (w, u and v). It is important to note that the in-plane displacements, u and v are functions of the three coordinates, x, y and z; whereas the deflection is only a function of the in-plane coordinates, x and y. The in-plane displacements are related to deflection as shown on equations 1 and 2 as:

$$u = -z\frac{dw}{dx} + u_0 \qquad (1)$$

$$v = -z\frac{dw}{dy} + v_0 \qquad (2)$$

Unlike isotropic plate, the middle surface in-plane displacements are not constants. The in-plane strains are the first derivatives of the in-plane displacements given as:

$$\varepsilon_{xx} = \frac{du}{dx} = \varepsilon_{xx}^0 + \varepsilon_{xx}^i = \frac{du_0}{dx} - z\frac{d^2w}{dx^2} \qquad (3)$$

$$\varepsilon_{yy} = \frac{dv}{dy} = \varepsilon_{yy}^0 + \varepsilon_{yy}^i = \frac{dv_0}{dy} - z\frac{d^2w}{dy^2} \qquad (4)$$

$$\gamma_{xy} = \varepsilon_{xy} + \varepsilon_{yx} = \left[-z\frac{d^2w}{dxdy} + \frac{du_0}{dy}\right] + \left[-z\frac{d^2w}{dxdy} + \frac{dv_0}{dx}\right]$$

That is:

- *Okoroafor S.U is currently pursuing PhD degree program in Structural Engineering in School of Postgraduate Studies Federal University of Technology Owerri, Nigeria, E-mail: Solomonokoroafory@mail.com*
- *Ezeh J.C is Professor of strcural Engineering in Civil engineering in Federal University Technology Owerri, Imo State Nigeria. E-mail: jcezeh2003e@mail.com*
(This information is optional; change it according to your need.)

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$\gamma_{xy} = \gamma_{xy}^{0} + \gamma_{xy}^{i} = \frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy} \qquad (5)$$

2.1 Constitutive relations for a lamina of thin laminated plate

The constitutive relation for a lamina of thin laminated plate is as shown in Equation 6:

$$\begin{bmatrix}\sigma_{11}\\ \sigma_{22}\\ \tau_{12}\end{bmatrix} = \begin{bmatrix}e_{11} & e_{12} & 0\\ e_{12} & e_{22} & 0\\ 0 & 0 & e_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{11}\\ \varepsilon_{22}\\ \gamma_{12}\end{bmatrix} \qquad (6)$$

$$Where: e_{11} = \frac{E_{11}}{1-\mu_{xy}\mu_{yx}};$$

$$e_{12} = \frac{E_{11}.\mu_{21}}{1-\mu_{xy}\mu_{yx}} = \frac{E_{22}.\mu_{12}}{1-\mu_{xy}\mu_{yx}};$$

$$e_{22} = \frac{E_{11}}{1-\mu_{xy}\mu_{yx}};\ e_{33} = G_{12}$$

E_{ij} and $\mu_{ij}$are the moduli of elasticity and Poisson's ratios of the anisotropic lamina. Equation 6 is transformed from the local coordinate (1-2 coordinate) system to global coordinate (x-y coordinate) system using the transformation matrix [T] as:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy2}\end{bmatrix} = [T]^{-1}\begin{bmatrix}e_{11} & e_{12} & 0\\ e_{12} & e_{22} & 0\\ 0 & 0 & e_{33}\end{bmatrix}[T]^{-T}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} \qquad (7)$$

Where; the transformation matrix, [T] is defined as:

$$[T] = \begin{bmatrix}m^2 & n^2 & 2mn\\ n^2 & m^2 & -2mn\\ -mn & mn & (m^2-n^2)\end{bmatrix} \qquad (8)$$

$$Where: \quad m = Cos\theta \ and \ n = Sin\theta$$

Substituting equation 8 into equation 7 gives:

$$\begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix} = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \gamma_{xy}\end{bmatrix} \qquad (9)$$

Where:

$$a_{11} = e_{11}m^4 + 2m^2n^2(e_{12}+2e_{33}) + e_{22}n^4$$

$$a_{12} = e_{12}m^4 + m^2n^2[e_{11}+e_{22}-4e_{33}] + e_{12}n^4$$

$$a_{13} = m^3n[e_{11}-e_{12}-2e_{33}] + mn^3[e_{12}-e_{22}+2e_{33}]$$

$$a_{21} = e_{12}[m^4+n^4] + m^2n^2[e_{11}+e_{22}-4e_{33}]$$

$$a_{22} = e_{22}m^4 + 2m^2n^2\,[e_{12}+2e_{33}] + e_{11}n^4$$

$$a_{23} = m^3n[e_{12}-e_{22}+2e_{33}] + mn^3[e_{11}-e_{12}-2f_{12}]$$

$$a_{31} = m^3n[e_{11}-e_{12}-2e_{33}] + mn^3[e_{12}-e_{22}+2e_{33}]$$

$$a_{32} = m^3n[e_{12}-e_{22}+2e_{33}] + mn^3[e_{11}-e_{12}-2e_{33}]$$

$$a_{33} = m^2n^2[e_{11}-2e_{12}+e_{22}-2e_{33}] + e_{33}[m^4+n^4]$$

Substituting equations 3, 4 and 5 into equation 9 gives:

$$[\sigma] = \begin{bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \tau_{xy}\end{bmatrix}$$

$$= \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\\ \frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\\ \frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\end{bmatrix} \qquad (10)$$

3.0 Total potential energy functional for a laminated thin rectangular plate

The total potential energy functional for a laminated thin rectangular plate is given as:

$$\Pi = \frac{1}{2}\iiint [\sigma][\varepsilon]\ dx\,.dy\,.dz - \iint\left(qw + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + \frac{m\lambda^2}{2}w^2\right)dx\,dy \qquad (12)$$

Substituting equations 10 and 11 into equation 12 gives Equation 13:

$$\Pi = \frac{1}{2}\iiint \begin{bmatrix}\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\\ \frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\\ \frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\end{bmatrix}^T \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\\ \frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\\ \frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\end{bmatrix} dxdydz - \iint\left(qw + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + \frac{m\lambda^2}{2}w^2\right)dx\,dy \qquad (13)$$

Carrying out the multiplication and closed domain integration of equation 13 with respect to z gives:

$$\Pi = \frac{1}{2}\iint\{\left(A_{11}\left[\frac{du_0}{dx}\right]^2 + 2A_{12}\frac{du_0}{dx}\frac{dv_0}{dy} + A_{33}\left[\frac{du_0}{dy}\right]^2 + 2A_{33}\frac{du_0}{dy}\frac{dv_0}{dx} + A_{33}\left[\frac{dv_0}{dx}\right]^2 + A_{22}\left[\frac{dv_0}{dy}\right]^2\right)$$

$$-2\left(B_{11}\frac{du_0}{dx}.\frac{d^2w}{dx^2} + (B_{12}+2B_{33})\frac{du_0}{dy}\frac{d^2w}{dxdy} + (B_{12}+2B_{33})\frac{dv_0}{dx}\frac{d^2w}{dxdy} + B_{22}\frac{dv_0}{dy}\frac{d^2w}{dy^2}\right)$$

$$+\left(D_{11}\left[\frac{d^2w}{dx^2}\right]^2 + 2(D_{12}+2D_{33})\left[\frac{d^2w}{dxdy}\right]^2 + D_{22}\left[\frac{d^2w}{dy^2}\right]^2\right)$$

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$+1.5\left(A_{13}\frac{du_0}{dx}\frac{du_0}{dy}+A_{13}\frac{du_0}{dx}\frac{dv_0}{dx}-3B_{13}\frac{du_0}{dx}\frac{d^2w}{dxdy}-B_{13}\frac{dv_0}{dx}\frac{d^2w}{dx^2}+2D_{13}\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$

$$+1.5\left(A_{23}\frac{du_0}{dy}\frac{dv_0}{dy}+A_{23}\frac{dv_0}{dy}\frac{dv_0}{dx}-3B_{23}\frac{dv_0}{dy}\frac{d^2w}{dxdy}-B_{23}\frac{du_0}{dy}\frac{d^2w}{dy^2}+2D_{23}\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)\Big\}\,dx\,.dy$$

$$-\iint\left(qw+\frac{N_x}{2}\left(\frac{dw}{dx}\right)^2+\frac{m\lambda^2}{2}w^2\right)dx\,dy \qquad (14)$$

If we assume m to stands for number of a lamina in the plate while n is the total number of laminas and:

$$A_{ij}=\sum_{m=1}^{m=n}a_{ij}(z_m-z_{m-1}) \qquad (15)$$

$$B_{ij}=\frac{1}{2}\sum_{m=1}^{m=n}a_{ij}(z_m^2-z_{m-1}^2) \qquad (16)$$

$$D_{ij}=\frac{1}{3}\sum_{m=1}^{m=n}a_{ij}(z_m^3-z_{m-1}^3) \qquad (17)$$

Let the summation of the following three constants be one. That is:

$$n_1+n_2+n_3=1 \qquad (18)$$

Substituting Equation 18 into Equation 14 gives Equation 19

$$\Pi=\Pi_1+\Pi_2+\Pi_3 \qquad (19)$$

$$\Pi_1=\frac{1}{2}\iint\Bigg[\Bigg(A_{11}\left[\frac{du_0}{dx}\right]^2+2A_{12}\frac{du_0}{dx}\frac{dv_0}{dy}+A_{33}\left[\frac{du_0}{dy}\right]^2+2A_{33}\frac{du_0}{dy}\frac{dv_0}{dx}+A_{33}\left[\frac{dv_0}{dx}\right]^2+A_{22}\left[\frac{dv_0}{dy}\right]^2\Bigg)-2\left(B_{11}\frac{du_0}{dx}\cdot\frac{d^2w}{dx^2}+(B_{12}+2B_{33})\frac{du_0}{dy}\frac{d^2w}{dxdy}+(B_{12}+2B_{33})\frac{dv_0}{dx}\frac{d^2w}{dxdy}+B_{22}\frac{dv_0}{dy}\frac{d^2w}{dy^2}\right)+\left(D_{11}\left[\frac{d^2w}{dx^2}\right]^2+2(D_{12}+2D_{33})\left[\frac{d^2w}{dxdy}\right]^2+D_{22}\left[\frac{d^2w}{dy^2}\right]^2\right)\Bigg]dRdQ\;-n_1\iint\left(qw+\frac{N_x}{2}\left(\frac{dw}{dx}\right)^2+\frac{m\lambda^2}{2}w^2\right)dx\,dy \qquad (20a)$$

$$\Pi_2=\frac{1.5}{2}\iint\left[A_{13}\frac{du_0}{dx}\frac{du_0}{dy}+A_{13}\frac{du_0}{dx}\frac{dv_0}{dx}-3B_{13}\frac{du_0}{dx}\frac{d^2w}{dxdy}-B_{13}\frac{dv_0}{dx}\frac{d^2w}{dx^2}+2D_{13}\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right]dx\,.dy$$

$$-n_2\iint\left(qw+\frac{N_x}{2}\left(\frac{dw}{dx}\right)^2+\frac{m\lambda^2}{2}w^2\right)dx\,dy \qquad (20b)$$

$$\Pi_3=\frac{1.5}{2}\iint\left[A_{23}\frac{du_0}{dy}\frac{dv_0}{dy}+A_{23}\frac{dv_0}{dy}\frac{dv_0}{dx}-3B_{23}\frac{dv_0}{dy}\frac{d^2w}{dxdy}-B_{23}\frac{du_0}{dy}\frac{d^2w}{dy^2}+2D_{23}\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right]dx\,.dy$$

$$-n_3\iint\left(qw+\frac{N_x}{2}\left(\frac{dw}{dx}\right)^2+\frac{m\lambda^2}{2}w^2\right)dx\,dy \qquad (20c)$$

The meaning for z, m and n for easy understanding is illustrated with the laminated plate of four laminas that is shown on Figure 1

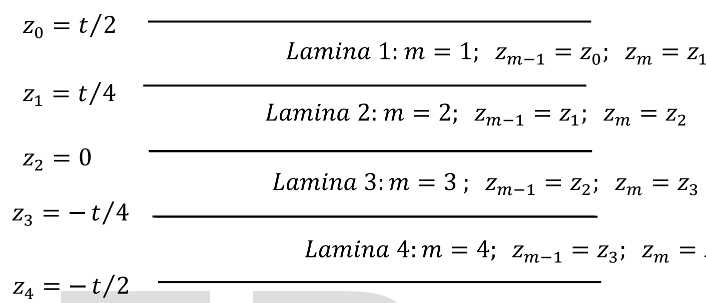

Figure 1: A laminated plate that is made of four laminas

3.1 General and direct Variation of Total potential energy functional for a laminated thin rectangular plate

Minimizing equations 20a, 20b and 20c with respect to w, u0 and v0 and making some rearrangements shall give the respective equations:

$$\frac{\partial\Pi_1}{\partial w}=0=\frac{1}{2}\iint\Bigg[-2\left(B_{11}\frac{\partial^3u_0}{\partial x^3}+(B_{12}+2B_{33})\frac{\partial^3u_0}{\partial x\partial y^2}+(B_{12}+2B_{33})\frac{\partial^3v_0}{\partial x^2\partial y}+B_{22}\frac{\partial^3v_0}{\partial y^3}\right)+2\left(D_{11}\frac{\partial^4w}{\partial x^4}+2(D_{12}+2D_{33})\frac{\partial^4w}{\partial x^2\partial y^2}+D_{22}\frac{\partial^4w}{\partial y^4}\right)\Bigg]dRdQ$$

$$-n_1\iint\left(q-N_x\frac{d^2w}{dx^2}+m\lambda^2w\right)dx\,dy \qquad (21a)$$

$$\frac{\partial\Pi_2}{\partial w}=0=0.75\iint\frac{\partial^2}{\partial x^2}\left(-3B_{13}\frac{\partial u_0}{\partial y}-B_{13}\frac{\partial v_0}{\partial x}+2D_{13}\frac{\partial^2w}{\partial x\partial y}\right)dx\,.dy$$

$$-n_2\iint\left(q-N_x\frac{d^2w}{dx^2}+m\lambda^2w\right)dx\,dy \qquad (21b)$$

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$\frac{\partial \Pi_3}{\partial \mathrm{w}} = 0 = \frac{1.5}{2}\iint \frac{\partial^2}{\partial y^2}\left(-3B_{23}\frac{\partial v_0}{\partial x} - B_{23}\frac{\partial u_0}{\partial y} + 2D_{23}\frac{\partial^2 w}{\partial x \partial y}\right)\, dx\,.dy$$

$$-n_3 \iint \left(q - N_x \frac{d^2 w}{dx^2} + m\lambda^2 w\right) dx\, dy \quad (21c)$$

$$\frac{\partial \Pi_1}{\partial u_0} = 0 = \iint \left(\frac{d^2}{dx^2}\left[A_{11}u_0 - B_{11}\frac{dw}{dx}\right] + \frac{d^2}{dxdy}\left[A_{12}v_0 - B_{12}\frac{dw}{dy}\right] + \frac{d^2}{dxdy}\left[A_{33}v_0 - B_{33}\frac{dw}{dy}\right] + \frac{d^2 u_0}{dy^2}\left[A_{33}u_0 - B_{33}\frac{dw}{dx}\right]\right) dRdQ \quad (22a)$$

$$\frac{\partial \Pi_2}{\partial u_0} = \frac{1.5}{2}\iint \left(2\frac{d^2}{dxdy}\left[A_{13}u_0 - B_{13}\frac{dw}{dx}\right] + \frac{d^2}{dx^2}\left[A_{13}v_0 - B_{13}\frac{dw}{dy}\right]\right)\, dx\,.dy = 0 \quad (22b)$$

$$\frac{\partial \Pi_3}{\partial u_0} = \frac{1.5}{2}\iint \frac{d^2}{dy^2}\left[A_{23}v_0 - B_{23}\frac{du_w}{dy}\right]\, dx\,.dy = 0 \quad (22c)$$

$$\frac{\partial \Pi_1}{\partial v_0} = \iint \left[\frac{d^2}{dxdy}\left(\left[A_{12}u_0 - B_{12}\frac{dw}{dx}\right] + \frac{d^2}{dy^2}\left[A_{22}v_0 - B_{22}\frac{dw}{dy}\right] + \frac{d^2}{dxdy}\left[A_{33}u_0 - B_{33}\frac{dw}{dx}\right] + \frac{d^2}{dx^2}\left[A_{33}v_0 - B_{33}\frac{dw}{dy}\right]\right)\right] dRdQ = 0 \quad (23a)$$

$$\frac{\partial \Pi_2}{\partial v_0} = \frac{1.5}{2}\iint \frac{d^2}{dx^2}\left[A_{13}u_0 - B_{13}\frac{dw}{dx}\right]\, dx\,.dy = 0 \quad (23b)$$

$$\frac{\partial \Pi_3}{\partial v_0} = \frac{1.5}{2}\iint \left(\frac{d^2}{dy^2}\left[A_{23}u_0 - B_{23}\frac{dw}{dx}\right] + 2\frac{d^2}{dxdy}\left[A_{23}v_0 - B_{23}\frac{dw}{dy}\right]\right)\, dx\,.dy = 0 \quad (23c)$$

For equations 22a, 22b, 22c, 23a, 23b and 23c to be true, the following shall hold (where c and d are yet to be determined constants):

$$u_0 = \frac{B_{ij}}{A_{ij}}\frac{\partial w}{\partial x} = c\frac{\partial w}{\partial x} \quad (24a)$$

$$v_0 = \frac{B_{ij}}{A_{ij}}\frac{\partial w}{\partial y} = d\frac{\partial w}{\partial y} \quad (24b)$$

Substituting equations 24a and 24b into equation 21a and making some rearrangements and observing that an integral can only be zero if its integrand gives:

$$\iint \left([D_{11} - cB_{11}]\frac{\partial^4 w}{\partial x^4} + 2[D_{12} - cB_{12} - dB_{12} + 2D_{33} - 2cB_{33} - 2dB_{33}]\frac{\partial^4 w}{\partial x^2 \partial y^2} + [D_{22} - dB_{22}]\frac{\partial^4 w}{\partial y^4}\right) dRdQ \;\; - n_1 \iint \left(q - N_x \frac{d^2 w}{dx^2} + m\lambda^2 w\right) dx\, dy = 0 \quad (25)$$

Dividing equation 25 by [D22 - dB22] gives:

$$\left[f_1\frac{\partial^4 w}{\partial x^4} + f_2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} - n_1\left(q - N_x\frac{d^2 w}{dx^2} + m\lambda^2 w\right)\right] = 0 \quad 26$$

$$Where\colon\; f_1 = \frac{[D_{11} - cB_{11}]}{[D_{22} - dB_{22}]};\; f_2 = \frac{2[D_{12} - cB_{12} - dB_{12} + 2D_{33} - 2cB_{33} - 2dB_{33}]}{[D_{22} - dB_{22}]}$$

The exact solution to equation 26 is in polynomial form is (see appendix A for details):

$$w = \beta_1 h \quad (27a)$$

$$w = \left(a_0 + a_1 x + \frac{a_2}{2!}x^2 + \frac{a_3}{3!}x^3 + \frac{a_4}{4!}x^4\right)\left(b_0 + b_1 y + \frac{b_2}{2!}y^2 + \frac{b_3}{3!}y^3 + \frac{b_4}{4!}y^4\right) \quad (27b)$$

Substituting equation 27a into equations 24a and 24b gives:

$$u_0 = c\beta_1\frac{\partial h}{\partial x} = \beta_2\frac{\partial h}{\partial x} \quad (28a)$$

$$v_0 = d\beta_1\frac{\partial h}{\partial y} = \beta_3\frac{\partial h}{\partial y} \quad (28b)$$

$$\beta_2 = c\beta_1 \;\; and \;\; \beta_3 = d\beta_1 \quad (28c)$$

Substituting equations 27a, 28a and 28b into equations 20a, 20b and 20c and writing the outcomes in terms of non dimensional coordinates gives:

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$\Pi_1 = \frac{ab}{2a^4}\iint\left[\left(A_{11}{\beta_2}^2\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{1}{\alpha^2}\left[2A_{12}\beta_2\beta_3 + 2A_{33}\beta_2\beta_3 + A_{33}{\beta_2}^2 + A_{33}{\beta_3}^2\right]\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 + A_{22}\frac{{\beta_3}^2}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right) - 2\left(B_{11}\beta_1\beta_2\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + (B_{12}+2B_{33})\frac{\beta_1\beta_2}{\alpha^2}\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 + (B_{12}+2B_{33})\frac{\beta_1\beta_3}{\alpha^2}\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 + B_{22}\frac{\beta_1\beta_3}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right) + \left(D_{11}{\beta_1}^2\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + 2(D_{12}+2D_{33})\frac{{\beta_1}^2}{\alpha^2}\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 + D_{22}\frac{{\beta_1}^2}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right)\right]dRdQ$$

$$-n_1 ab\iint\left(q\beta_1 h + \frac{N_x}{2a^2}{\beta_1}^2\left(\frac{dh}{dR}\right)^2 + {\beta_1}^2\frac{m\lambda^2}{2}h^2\right)dR\,dQ \qquad (29a)$$

$$\Pi_2 = \frac{1.5ab}{2a^4}\iint\left[A_{13}\frac{{\beta_2}^2}{\alpha}\frac{\partial^2 h}{\partial R^2}.\frac{\partial^2 h}{\partial R\partial Q} + A_{13}\frac{\beta_2\beta_3}{\alpha}\frac{\partial^2 h}{\partial R^2}.\frac{\partial^2 h}{\partial R\partial Q} - 3B_{13}\frac{\beta_1\beta_2}{\alpha}\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q} - B_{13}\frac{\beta_1\beta_3}{\alpha}.\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q} + 2D_{13}\frac{{\beta_1}^2}{\alpha}\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q}\right]dR\,.dQ - n_2 ab\iint\left(q\beta_1 h + \frac{N_x}{2a^2}{\beta_1}^2\left(\frac{dh}{dR}\right)^2 + {\beta_1}^2\frac{m\lambda^2}{2}h^2\right)dR\,dQ \qquad (29b)$$

$$\Pi_3 = \frac{1.5ab}{2a^4}\iint\left[A_{23}\frac{\beta_2\beta_3}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} + A_{23}\frac{{\beta_3}^2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} - 3B_{23}\frac{\beta_1\beta_3}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} - B_{23}\frac{\beta_1\beta_2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} + 2D_{23}\frac{{\beta_1}^2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2}\right]dR\,.dQ - n_3 ab\iint\left(q\beta_1 h + \frac{N_x}{2a^2}{\beta_1}^2\left(\frac{dh}{dR}\right)^2 + {\beta_1}^2\frac{m\lambda^2}{2}h^2\right)dR\,dQ \qquad (29c)$$

Minimizing equations 29a, 29b and 29c with respect to β1 and rearrange gives respectively:

$$\frac{d\Pi_1}{d\beta_1} = 0 = -\left(B_{11}\beta_2 k_x + (B_{12}+2B_{33})\frac{\beta_2}{\alpha^2}k_{xy} + (B_{12}+2B_{33})\frac{\beta_3}{\alpha^2}k_{xy} + B_{22}\frac{\beta_3}{\alpha^4}k_y\right) + \beta_1\left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y\right) - n_1 a^4\left(qk_q + \frac{N_x}{a^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda\right) \quad 30a$$

$$\frac{d\Pi_2}{d\beta_1} = 0 = \left(3D_{13}\frac{\beta_1}{\alpha} - 2.25B_{13}\frac{\beta_2}{\alpha} - 0.75B_{13}\frac{\beta_3}{\alpha}\right)k_{xxy} - n_2 a^4\left(qk_x + \frac{N_x}{a^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda\right) \quad 30b$$

$$\frac{d\Pi_3}{d\beta_1} = 0 = \left(3D_{23}\frac{\beta_1}{\alpha^3} - 0.75B_{23}\frac{\beta_2}{\alpha^3} - 2.25B_{23}\frac{\beta_3}{\alpha^3}\right)k_{xyy} - n_3 a^4\left(qk_x + \frac{N_x}{a^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda\right) \quad (30c)$$

Minimizing equations 29a, 29b and 29c with respect to β2 gives:

$$\frac{d\Pi_1}{d\beta_2} = \frac{ab}{\alpha^4}\left[\left(A_{11}\beta_2 k_x + \frac{1}{\alpha^2}[A_{12}\beta_3 + A_{33}\beta_2 + A_{33}\beta_3]k_{xy}\right) - B_{11}\beta_1 k_x - (B_{12}+2B_{33})\frac{\beta_1}{\alpha^2}k_{xy}\right] = 0 \quad 31a$$

$$\frac{d\Pi_2}{d\beta_2} = \frac{ab}{\alpha^4}\left[1.5A_{13}\frac{\beta_2}{\alpha}k_{xxy} + 0.75A_{13}\frac{\beta_3}{\alpha}k_{xxy} - 2.25B_{13}\frac{\beta_1}{\alpha}k_{xxy}\right] = 0 \quad 31b$$

$$\frac{d\Pi_3}{d\beta_2} = 0 = \frac{ab}{\alpha^4}\left[0.75A_{23}\frac{\beta_3}{\alpha^3}k_{xyy} - 0.75B_{23}\frac{\beta_1}{\alpha^3}k_{xyy}\right] \quad 31c$$

Minimizing equations 29a, 29b and 29c with respect to β3 gives:

$$\frac{d\Pi_1}{d\beta_3} = \frac{ab}{\alpha^4}\left[A_{22}\frac{\beta_3}{\alpha^4}k_y + \frac{1}{\alpha^2}[A_{12}\beta_2 + A_{33}\beta_2 + A_{33}\beta_3]k_{xy} - B_{22}\frac{\beta_1}{\alpha^4}k_y - (B_{12}+2B_{33})\frac{\beta_1}{\alpha^2}k_{xy}\right] = 0 \quad (32a)$$

$$\frac{d\Pi_2}{d\beta_3} = 0 = \frac{ab}{\alpha^4}\left[0.75A_{13}\frac{\beta_2}{\alpha}k_{xxy} - 0.75B_{13}\frac{\beta_1}{\alpha}.k_{xxy}\right] \quad (32b)$$

$$\frac{d\Pi_3}{d\beta_3} = 0 = \frac{ab}{\alpha^4}\left[0.75A_{23}\frac{\beta_2}{\alpha^3}k_{xyy} + 1.5A_{23}\frac{\beta_3}{\alpha^3}k_{xyy} - 2.25B_{23}\frac{\beta_1}{\alpha^3}k_{xyy}\right] \quad (32c)$$

$$Where: k_x = \iint\left(\frac{\partial^2 h}{\partial R^2}\right)^2 dR\,.dQ \;:\; k_{xy} = \iint\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 dR\,.dQ \;:\; k_y = \iint\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dR\,.dQ$$

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$k_{xxy} = \iint \frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R \partial Q}\ \ dR\,.dQ : k_{xyy} = \iint \frac{\partial^2 h}{\partial R \partial Q}.\frac{\partial^2 h}{\partial Q^2}\ \ dR\,.dQ\ \ :\ k_q = \iint h\ \ dR\,.dQ$$

$$k_N = \iint \left(\frac{dh}{dR}\right)^2\ \ dR\,.dQ\ :\ k_\lambda = \iint h^2\ \ dR\,.dQ$$

Adding the equations 30a, 30b and 30c together and rearranging the outcome gives:

$$\frac{d\Pi}{d\beta_1} = \frac{d\Pi_1}{d\beta_1} + \frac{d\Pi_2}{d\beta_1} + \frac{d\Pi_3}{d\beta_1} = 0$$
$$= \beta_1\left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y + 3\frac{D_{13}}{\alpha}k_{xxy} + 3\frac{D_{23}}{\alpha^3}k_{xyy}\right) - \beta_2\left(B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 2.25B_{13}\frac{k_{xxy}}{\alpha} + 0.75B_{23}\frac{k_{xyy}}{\alpha^3}\right) - \beta_3\left((B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + B_{22}\frac{k_y}{\alpha^4} + 0.75B_{13}\frac{k_{xxy}}{\alpha} + 2.25B_{23}\frac{k_{xyy}}{\alpha^3}\right) - a^4(n_1+n_2+n_3)\left(qk_x + \frac{N_x}{\alpha^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda\right) \qquad (33a)$$

Substituting equation 18 into equation 33a and rearranging the outcome gives:

$$a^4\left(qk_q + \frac{N_x}{\alpha^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda\right) = \beta_1\left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y + 3\frac{D_{13}}{\alpha}k_{xxy} + 3\frac{D_{23}}{\alpha^3}k_{xyy}\right) - \beta_2\left(B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 2.25B_{13}\frac{k_{xxy}}{\alpha} + 0.75B_{23}\frac{k_{xyy}}{\alpha^3}\right) - \beta_3\left((B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + B_{22}\frac{k_y}{\alpha^4} + 0.75B_{13}\frac{k_{xxy}}{\alpha} + 2.25B_{23}\frac{k_{xyy}}{\alpha^3}\right) \qquad (33b)$$

Adding the equations 31a, 31b and 31c together and rearranging the outcome gives:

$$\beta_2\left(A_{11}k_x + A_{33}\frac{k_{xy}}{\alpha^2} + 1.5A_{13}\frac{k_{xxy}}{\alpha}\right) + \beta_3\left(A_{12}\frac{k_{xy}}{\alpha^2} + A_{33}\frac{k_{xy}}{\alpha^2} + 0.75A_{13}\frac{k_{xxy}}{\alpha} + 0.75A_{23}\frac{k_{xyy}}{\alpha^3}\right) = \beta_1\left(B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 2.25B_{13}\frac{k_{xxy}}{\alpha} + 0.75B_{23}\frac{k_{xyy}}{\alpha^3}\right) \qquad (34a)$$

Adding the equations 32a, 32b and 32c together and rearranging the outcome gives:

$$\beta_2\left(A_{12}\frac{k_{xy}}{\alpha^2} + A_{33}\frac{k_{xy}}{\alpha^2} + 0.75A_{13}\frac{k_{xxy}}{\alpha} + 0.75A_{23}\frac{k_{xyy}}{\alpha^3}\right) + \beta_3\left(A_{22}\frac{k_y}{\alpha^4} + A_{33}\frac{k_{xy}}{\alpha^2} + 1.5A_{23}\frac{k_{xyy}}{\alpha^3}\right) = \beta_1\left(B_{22}\frac{k_y}{\alpha^4} + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 0.75B_{13}\frac{k_{xxy}}{\alpha} + 2.25B_{23}\frac{k_{xyy}}{\alpha^3}\right) \qquad (34b)$$

Solving equations 34a and 34b simultaneously gives:

$$\beta_2 = T_2\beta_1 = \beta_1\frac{(d_{12}.d_{23} - d_{13}.d_{22})}{({d_{12}}^2 - d_{11}d_{22})} \qquad (35a)$$

$$\beta_3 = T_3\beta_1 = \beta_1\frac{(d_{12}.d_{13} - d_{11}d_{23})}{({d_{12}}^2 - d_{11}d_{22})} \qquad (35b)$$

Where:

$$d_{11} = A_{11}k_x + A_{33}\frac{k_{xy}}{\alpha^2} + 1.5A_{13}\frac{k_{xxy}}{\alpha} \qquad (36a)$$

$$d_{12} = A_{12}\frac{k_{xy}}{\alpha^2} + A_{33}\frac{k_{xy}}{\alpha^2} + 0.75A_{13}\frac{k_{xxy}}{\alpha} + 0.75A_{23}\frac{k_{xyy}}{\alpha^3} \qquad (36b)$$

$$d_{22} = A_{22}\frac{k_y}{\alpha^4} + A_{33}\frac{k_{xy}}{\alpha^2} + 1.5A_{23}\frac{k_{xyy}}{\alpha^3} \qquad (36c)$$

$$d_{13} = B_{11}k_x + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 2.25B_{13}\frac{k_{xxy}}{\alpha} + 0.75B_{23}\frac{k_{xyy}}{\alpha^3} \qquad (36d)$$

$$d_{23} = B_{22}\frac{k_y}{\alpha^4} + (B_{12}+2B_{33})\frac{k_{xy}}{\alpha^2} + 0.75B_{13}\frac{k_{xxy}}{\alpha} + 2.25B_{23}\frac{k_{xyy}}{\alpha^3} \qquad (36e)$$

Substituting equations 35a and 35b into equation 33b and rearranging gives:

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$\left(\frac{qa^4}{\beta_1}k_q + N_x a^2 k_N + a^4 m\lambda^2 k_\lambda\right) = \left(D_{11}k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33})k_{xy} + \frac{D_{22}}{\alpha^4}k_y + 3\frac{D_{13}}{\alpha}k_{xxy} + 3\frac{D_{23}}{\alpha^3}k_{xyy}\right) - T_2\left(B_{11}k_x + \frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{2.25B_{13}}{\alpha}k_{xxy} + \frac{0.75B_{23}}{\alpha^3}k_{xyy}\right) - T_3\left(\frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{B_{22}}{\alpha^4}k_y + \frac{0.75B_{13}}{\alpha}k_{xxy} + \frac{2.25B_{23}}{\alpha^3}k_{xyy}\right) \quad (37a)$$

The equation for obtaining the exact solution of thin rectangular laminate plate is obtained by
Dividing equation 37a by D22 which gives:

$$\left(\frac{qa^4}{D_{22}\beta_1}k_q + \frac{N_x a^2}{D_{22}}k_N + \frac{a^4 m\lambda^2}{D_{22}}k_\lambda\right) = k_{T1} + k_{T2} + k_{T3} \quad (37b)$$

Where:

$$k_{T1} = \left(\emptyset_{11}k_x + \frac{2\emptyset_{12}}{\alpha^2}k_{xy} + \frac{1}{\alpha^4}k_y + \frac{3\emptyset_{13}}{\alpha}k_{xxy} + \frac{3\emptyset_{23}}{\alpha^3}k_{xyy}\right) \quad (38a)$$

$$k_{T2} = -\frac{T_2}{D_{22}}\left(B_{11}k_x + \frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{2.25B_{13}}{\alpha}k_{xxy} + \frac{0.75B_{23}}{\alpha^3}k_{xyy}\right) \quad (38b)$$

$$k_{T3} = -\frac{T_3}{D_{22}}\left(\frac{(B_{12}+2B_{33})}{\alpha^2}k_{xy} + \frac{B_{22}}{\alpha^4}k_y + \frac{0.75B_{13}}{\alpha}k_{xxy} + \frac{2.25B_{23}}{\alpha^3}k_{xyy}\right) \quad (38c)$$

$$\emptyset_{11} = \frac{D_{11}}{D_{22}}; \emptyset_{12} = \frac{D_{12}+2D_{33}}{D_{22}}; \emptyset_{13} = \frac{D_{13}}{D_{22}}; \emptyset_{23} = \frac{23}{D_{22}} \quad (38d)$$

Equation 37b can rearranged and written distinctly for cases of pure bending, buckling and free vibration as:

$$\frac{D_{22}\beta_1}{qa^4} = \frac{k_q}{k_{T1}+k_{T2}+k_{T3}} \quad (39a)$$

$$\frac{N_x a^2}{D_{22}} = \frac{k_{T1}+k_{T2}+k_{T3}}{k_N} \quad (39b)$$

$$\frac{a^4 m\lambda^2}{D_{22}} = \frac{k_{T1}+k_{T2}+k_{T3}}{k_\lambda} \quad (39c)$$

4.0 Results and Discussions

4.1 Results

Equation 3.39a 3.39b and 3.39c stand as equation for obtaining the pure bending, buckling and free vibration of rectangular thin laminated plates for any number of layers.

NUMERICAL EXAMPLES

For a laminated simply supported member in all the edges with the following properties: - four-layer laminas, orientation – 0/90/90/0; E1/E2 = 25; G12/E2 = 0.5; V12 = 0.25.

	k_x	k_{xy}	k_y	k_{xxy}	k_{xyy}	k_L	p	k_q	k_N
ss ss	0.23 621	0.23 591	0.23 621	0	0	0.002 421	1	0. 04	0.02 39

The buckling values is as shown Table 1 and Table 2;

Table 4.1: Buckling load using the simple approach

PRESENT WORK					
0/90/90/0			0/90/90/0		
E1/E2 = 25; G12/E2 = 0.5; V12 = 0.25			E1/E2 = 25; G12/E2 = 0.5; V12 = 0.25		
P = b/a	Nx. a2/D22	Divide by (pi)2	P = a/b	Nx. b2/(p2. D22)	Divide by (pi)2
0.6666667	118.245	11.981	1.5	52.553	5.325
1	70.398	7.133	1	70.398	7.133
2	56.515	5.726	0.5	226.059	22.905

Table 4.2: Buckling load obtained by Reddy; Osman and suleiman

REDDY J.N (2004)			M. Y. Osman and O. M. E. Suleiman (2017)		
0/90/90/0			0/90/90/0		
E1/E2 = 25; G12/E2 = 0.5; V12 = 0.25			E1/E2 = 25; G12/E2 = 0.5; V12 = 0.25		
P = a/b	Nx. $b^2/(p^2$. D22)	Divide by $(pi)^2$	P = a/b	Nx. b2/(p2. D22)	Divide by (pi)2
1.5	52.487	5.318	1.5	51.529	5.221
1	70.311	7.124	1	69.117	7.003
0.5	225.757	22.874	0.5	222.717	22.566

For a laminated simply supported member in all the edges with the following properties: - four-layer laminas, orientation – 0/90/90/0; G12/E2 = 0.5; V12 = 0.25. The natural free vibration λ for different aspect ratio and different E1/E2 is as shown Table 4.3 and Table 4. 4.

Table 4.3a: Natural Frequency using the simple approach

PRESENT				
G12/E2 = 0.5; V12 = 0.25				
0/90/90/0	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$
p = b/a	E1/E2 = 10	E1/E2 = 20	E1/E2 = 30	E1/E2 = 40
2	2.130	2.341	2.431	2.481
1	2.521	2.640	2.693	2.723
0.66667	3.448	3.459	3.464	3.466

0.5	4.989	4.921	4.888	4.870
0.4	7.115	6.998	6.943	6.912
0.333333	9.790	9.644	9.576	9.537

Table 4.3b: Natural Frequency using the simple approach but presented in Reddy form

Present work converted from λa^2 to λb^2				
G12/E2 = 0.5; V12 = 0.25				
0/90/90/0	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$
p = b/a	E1/E2 = 10	E1/E2 = 20	E1/E2 = 30	E1/E2 = 40
2	8.522	9.362	9.724	9.926
1	2.521	2.640	2.693	2.723
0.66667	1.532	1.537	1.539	1.541
0.5	1.247	1.230	1.222	1.217
0.4	1.138	1.120	1.111	1.106
0.33333	1.088	1.072	1.064	1.060

Table 4.4: Natural Frequency obtained by Reddy

REDDY J.N (2004)				
G12/E2 = 0.5; V12 = 0.25				
0/90/90/0	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$	$\lambda/(\pi^2)$
p = a/b	E1/E2 = 10	E1/E2 = 20	E1/E2 = 30	E1/E2 = 40
0.5	8.515	9.355	9.716	9.917
1	2.519	2.638	2.691	2.721
1.5	1.531	1.536	1.538	1.539
2	1.246	1.229	1.221	1.216
2.5	1.138	1.119	1.11	1.105
3	1.087	1.071	1.063	1.059

4.2 Discussion of Results

The results obtained from the Simple approach to analysis of thin laminated rectangular plates with exact displacement functions for buckling of SSSS plate, when compared with the work of Reddy J.N (2004) and that of M. Y. Osman and O. M. E. Suleiman (2017) shows that the percentage difference is less than 0.1% and when the results obtained from the approach for free vibration were compared with the work of Reddy J.N (2004), the results shows that the percentage difference is less than 0.06%.

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International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

APPENDIX

$$\Pi = \frac{1}{2}\iiint \begin{bmatrix} \frac{du_0}{dx} - z\frac{d^2w}{dx^2} \\ \frac{dv_0}{dy} - z\frac{d^2w}{dy^2} \\ \frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy} \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} \frac{du_0}{dx} - z\frac{d^2w}{dx^2} \\ \frac{dv_0}{dy} - z\frac{d^2w}{dy^2} \\ \frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy} \end{bmatrix} dx\,.dy\,.dz$$

$$- \iint \left(qw + \frac{N_x}{2}\left(\frac{dw}{dx}\right)^2 + \frac{m\lambda^2}{2}w^2\right) dx\,dy \qquad (13)$$

Carrying out the multiplication and closed domain integration of equation 13 with respect to z gives:

$$a_{11}\left(\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\right)^2 + a_{21}\left(\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\right)\left(\frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\right)$$
$$+ a_{31}\left(\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\right)\left(\frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\right)$$
$$+ a_{12}\left(\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\right)\left(\frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\right)$$
$$+ a_{22}\left(\frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\right)^2$$
$$+ a_{32}\left(\frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\right)\left(\frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\right)$$
$$+ a_{13}\left(\frac{du_0}{dx} - z\frac{d^2w}{dx^2}\right)\left(\frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\right)$$
$$+ a_{23}\left(\frac{dv_0}{dy} - z\frac{d^2w}{dy^2}\right)\left(\frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\right)$$
$$+ a_{33}\left(\frac{du_0}{dy} + \frac{dv_0}{dx} - 2z\frac{d^2w}{dxdy}\right)^2$$

$$a_{11}\left(\left[\frac{du_0}{dx}\right]^2 - 2z\frac{du_0}{dx}.\frac{d^2w}{dx^2} + z^2\left[\frac{d^2w}{dx^2}\right]^2\right)$$
$$+ a_{21}\left(\frac{du_0}{dx}\frac{dv_0}{dy} - z\frac{du_0}{dx}\frac{d^2w}{dy^2} - z\frac{dv_0}{dy}\frac{d^2w}{dx^2} + z^2\left[\frac{d^2w}{dxdy}\right]^2\right)$$
$$+ a_{31}\left(\frac{du_0}{dx}\frac{du_0}{dy} + \frac{du_0}{dx}\frac{dv_0}{dx} - 2z\frac{du_0}{dx}\frac{d^2w}{dxdy} - z\frac{d^2w}{dx^2}\frac{du_0}{dy} - z\frac{d^2w}{dx^2}\frac{dv_0}{dx} + 2z^2\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$
$$+ a_{12}\left(\frac{du_0}{dx}\frac{dv_0}{dy} - z\frac{du_0}{dx}\frac{d^2w}{dy^2} - z\frac{dv_0}{dy}\frac{d^2w}{dx^2} + z^2\frac{d^2w}{dx^2}\frac{d^2w}{dy^2}\right)$$
$$+ a_{22}\left(\left[\frac{dv_0}{dy}\right]^2 - 2z\frac{dv_0}{dy}\frac{d^2w}{dy^2} + z^2\left[\frac{d^2w}{dy^2}\right]^2\right)$$
$$+ a_{32}\left(\frac{dv_0}{dy}\frac{du_0}{dy} + \frac{dv_0}{dy}\frac{dv_0}{dx} - 2z\frac{dv_0}{dy}\frac{d^2w}{dxdy} - z\frac{d^2w}{dy^2}\frac{du_0}{dy} - z\frac{d^2w}{dy^2}\frac{dv_0}{dx} + 2z^2\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)$$
$$+ a_{13}\left(\frac{du_0}{dx}\frac{du_0}{dy} + \frac{du_0}{dx}\frac{dv_0}{dx} - 2z\frac{du_0}{dx}\frac{d^2w}{dxdy} - z\frac{d^2w}{dx^2}\frac{du_0}{dy} - z\frac{d^2w}{dx^2}\frac{dv_0}{dx} + 2z^2\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$
$$+ a_{23}\left(\frac{dv_0}{dy}\frac{du_0}{dy} + \frac{dv_0}{dy}\frac{dv_0}{dx} - 2z\frac{dv_0}{dy}\frac{d^2w}{dxdy} - z\frac{d^2w}{dy^2}\frac{du_0}{dy} - z\frac{d^2w}{dy^2}\frac{dv_0}{dx} + 2z^2\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)$$
$$+ a_{33}\left(\left[\frac{du_0}{dy}\right]^2 + 2\frac{du_0}{dy}\frac{dv_0}{dx} - 4z\frac{du_0}{dy}\frac{d^2w}{dxdy} + \left[\frac{dv_0}{dx}\right]^2 - 2\frac{dv_0}{dx}\frac{d^2w}{dxdy} + 4z^2\left[\frac{d^2w}{dxdy}\right]^2\right)$$

$$a_{11}\left(\left[\frac{du_0}{dx}\right]^2 - 2z\frac{du_0}{dx}.\frac{d^2w}{dx^2} + z^2\left[\frac{d^2w}{dx^2}\right]^2\right)$$
$$+ a_{21}\left(\frac{du_0}{dx}\frac{dv_0}{dy} - z\frac{du_0}{dx}\frac{d^2w}{dy^2} - z\frac{dv_0}{dy}\frac{d^2w}{dx^2} + z^2\left[\frac{d^2w}{dxdy}\right]^2\right)$$
$$+ a_{31}\left(\frac{du_0}{dx}\frac{du_0}{dy} + \frac{du_0}{dx}\frac{dv_0}{dx} - 2z\frac{du_0}{dx}\frac{d^2w}{dxdy} - z\frac{d^2w}{dx^2}\frac{du_0}{dy} - z\frac{d^2w}{dx^2}\frac{dv_0}{dx} + 2z^2\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$
$$+ a_{12}\left(\frac{du_0}{dx}\frac{dv_0}{dy} - z\frac{du_0}{dx}\frac{d^2w}{dy^2} - z\frac{dv_0}{dy}\frac{d^2w}{dx^2} + z^2\frac{d^2w}{dx^2}\frac{d^2w}{dy^2}\right)$$

$$+ a_{22}\left(\left[\frac{dv_0}{dy}\right]^2 - 2z\frac{dv_0}{dy}\frac{d^2w}{dy^2} + z^2\left[\frac{d^2w}{dy^2}\right]^2\right)$$

$$+ a_{23}\left(\frac{dv_0}{dy}\frac{du_0}{dy} + \frac{dv_0}{dy}\frac{dv_0}{dx} - 2z\frac{dv_0}{dy}\frac{d^2w}{dxdy} - z\frac{d^2w}{dy^2}\frac{du_0}{dy} - z\frac{d^2w}{dy^2}\frac{dv_0}{dx} + 2z^2\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)$$
$$+ a_{33}\left(\left[\frac{du_0}{dy}\right]^2 + 2\frac{du_0}{dy}\frac{dv_0}{dx} - 4z\frac{du_0}{dy}\frac{d^2w}{dxdy} + \left[\frac{dv_0}{dx}\right]^2 - 2\frac{dv_0}{dx}\frac{d^2w}{dxdy} + 4z^2\left[\frac{d^2w}{dxdy}\right]^2\right)$$

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$+a_{32}\left(\frac{dv_0}{dy}\frac{du_0}{dy}+\frac{dv_0}{dy}\frac{dv_0}{dx}-2z\frac{dv_0}{dy}\frac{d^2w}{dxdy}-z\frac{d^2w}{dy^2}\frac{du_0}{dy}-z\frac{d^2w}{dy^2}\frac{dv_0}{dx}+2z^2\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)$$

$$+a_{13}\left(\frac{du_0}{dx}\frac{du_0}{dy}+\frac{du_0}{dx}\frac{dv_0}{dx}-2z\frac{du_0}{dx}\frac{d^2w}{dxdy}-z\frac{d^2w}{dx^2}\frac{du_0}{dy}-z\frac{d^2w}{dx^2}\frac{dv_0}{dx}+2z^2\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$

$$\left(A_{11}\left[\frac{du_0}{dx}\right]^2-2B_{11}\frac{du_0}{dx}.\frac{d^2w}{dx^2}+D_{11}\left[\frac{d^2w}{dx^2}\right]^2\right)+\left(2A_{12}\frac{du_0}{dx}\frac{dv_0}{dy}-2B_{12}\frac{du_0}{dx}\frac{d^2w}{dy^2}-2B_{12}\frac{dv_0}{dy}\frac{d^2w}{dx^2}+2D_{12}\left[\frac{d^2w}{dxdy}\right]^2\right)$$
$$+\left(A_{22}\left[\frac{dv_0}{dy}\right]^2-2B_{22}\frac{dv_0}{dy}\frac{d^2w}{dy^2}+D_{22}\left[\frac{d^2w}{dy^2}\right]^2\right)$$
$$+\left(1.5A_{13}\frac{du_0}{dx}\frac{du_0}{dy}+1.5A_{13}\frac{du_0}{dx}\frac{dv_0}{dx}-3B_{13}\frac{du_0}{dx}\frac{d^2w}{dxdy}-1.5B_{13}\frac{du_0}{dy}\frac{d^2w}{dx^2}-1.5B_{13}\frac{dv_0}{dx}\frac{d^2w}{dx^2}+3D_{13}\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}\right)$$
$$+\left(1.5A_{23}\frac{du_0}{dy}\frac{dv_0}{dy}+1.5A_{23}\frac{dv_0}{dy}\frac{dv_0}{dx}-3B_{23}\frac{dv_0}{dy}\frac{d^2w}{dxdy}-1.5B_{23}\frac{du_0}{dy}\frac{d^2w}{dy^2}-1.5B_{23}\frac{dv_0}{dx}\frac{d^2w}{dy^2}+3D_{23}\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}\right)$$
$$+\left(A_{33}\left[\frac{du_0}{dy}\right]^2+2A_{33}\frac{du_0}{dy}\frac{dv_0}{dx}-4B_{33}\frac{du_0}{dy}\frac{d^2w}{dxdy}+A_{33}\left[\frac{dv_0}{dx}\right]^2-4B_{33}\frac{dv_0}{dx}\frac{d^2w}{dxdy}+4D_{33}\left[\frac{d^2w}{dxdy}\right]^2\right)$$

$$\left(A_{11}\left[\frac{du_0}{dx}\right]^2+2A_{12}\frac{du_0}{dx}\frac{dv_0}{dy}+A_{33}\left[\frac{du_0}{dy}\right]^2+2A_{33}\frac{du_0}{dy}\frac{dv_0}{dx}+A_{33}\left[\frac{dv_0}{dx}\right]^2+A_{22}\left[\frac{dv_0}{dy}\right]^2\right)$$
$$+1.5\left(A_{13}\frac{du_0}{dx}\frac{du_0}{dy}+A_{13}\frac{du_0}{dx}\frac{dv_0}{dx}+A_{23}\frac{du_0}{dy}\frac{dv_0}{dy}+A_{23}\frac{dv_0}{dy}\frac{dv_0}{dx}\right)$$

$$-2\left(B_{11}\frac{du_0}{dx}.\frac{d^2w}{dx^2}+B_{12}\frac{du_0}{dy}\frac{d^2w}{dxdy}+2B_{33}\frac{du_0}{dy}\frac{d^2w}{dxdy}+B_{12}\frac{dv_0}{dx}\frac{d^2w}{dxdy}+2B_{33}\frac{dv_0}{dx}\frac{d^2w}{dxdy}+B_{22}\frac{dv_0}{dy}\frac{d^2w}{dy^2}\right)$$

$$-1.5\left(2B_{13}\frac{du_0}{dx}\frac{d^2w}{dxdy}+B_{13}\frac{du_0}{dx}\frac{d^2w}{dxdy}+B_{13}\frac{dv_0}{dx}\frac{d^2w}{dx^2}\right)-1.5\left(2B_{23}\frac{dv_0}{dy}\frac{d^2w}{dxdy}+B_{23}\frac{du_0}{dy}\frac{d^2w}{dy^2}+B_{23}\frac{dv_0}{dy}\frac{d^2w}{dxdy}\right)$$
$$+\left(D_{11}\left[\frac{d^2w}{dx^2}\right]^2+2D_{12}\left[\frac{d^2w}{dxdy}\right]^2+4D_{33}\left[\frac{d^2w}{dxdy}\right]^2+D_{22}\left[\frac{d^2w}{dy^2}\right]^2\right)+3D_{13}\frac{d^2w}{dx^2}\frac{d^2w}{dxdy}+3D_{23}\frac{d^2w}{dy^2}\frac{d^2w}{dxdy}$$

Substituting equations 27a, 28a and 28b into equations 20a, 20b and 20c and writing the outcomes in terms of non dimensional coordinates gives

$$\Pi_1=\frac{ab}{2a^4}\iint\left[\left(A_{11}{\beta_2}^2\left(\frac{\partial^2h}{\partial R^2}\right)^2+\frac{1}{\alpha^2}\left[2A_{12}\beta_2\beta_3+2A_{33}\beta_2\beta_3+A_{33}{\beta_2}^2+A_{33}{\beta_3}^2\right]\left(\frac{\partial^2h}{\partial R\partial Q}\right)^2+A_{22}\frac{{\beta_3}^2}{\alpha^4}\left(\frac{\partial^2h}{\partial Q^2}\right)^2\right)\right.$$
$$-2\left(B_{11}\beta_1\beta_2\left(\frac{\partial^2h}{\partial R^2}\right)^2+(B_{12}+2B_{33})\frac{\beta_1\beta_2}{\alpha^2}\left(\frac{\partial^2h}{\partial R\partial Q}\right)^2+(B_{12}+2B_{33})\frac{\beta_1\beta_3}{\alpha^2}\left(\frac{\partial^2h}{\partial R\partial Q}\right)^2+B_{22}\frac{\beta_1\beta_3}{\alpha^4}\left(\frac{\partial^2h}{\partial Q^2}\right)^2\right)$$
$$\left.+\left(D_{11}{\beta_1}^2\left(\frac{\partial^2h}{\partial R^2}\right)^2+2(D_{12}+2D_{33})\frac{{\beta_1}^2}{\alpha^2}\left(\frac{\partial^2h}{\partial R\partial Q}\right)^2+D_{22}\frac{{\beta_1}^2}{\alpha^4}\left(\frac{\partial^2h}{\partial Q^2}\right)^2\right)\right]dRdQ$$

$$-n_1ab\iint\left(q\beta_1h+\frac{N_x}{2a^2}{\beta_1}^2\left(\frac{dh}{dR}\right)^2+{\beta_1}^2\frac{m\lambda^2}{2}h^2\right)dR\,dQ \quad (29a)$$

$$\Pi_2=\frac{1.5ab}{2a^4}\iint\left[A_{13}\frac{{\beta_2}^2}{\alpha}\frac{\partial^2h}{\partial R^2}.\frac{\partial^2h}{\partial R\partial Q}+A_{13}\frac{\beta_2\beta_3}{\alpha}\frac{\partial^2h}{\partial R^2}.\frac{\partial^2h}{\partial R\partial Q}-3B_{13}\frac{\beta_1\beta_2}{\alpha}\frac{\partial^2h}{\partial R^2}\frac{\partial^2h}{\partial R\partial Q}-B_{13}\frac{\beta_1\beta_3}{\alpha}.\frac{\partial^2h}{\partial R^2}\frac{\partial^2h}{\partial R\partial Q}+2D_{13}\frac{{\beta_1}^2}{\alpha}\frac{\partial^2h}{\partial R^2}\frac{\partial^2h}{\partial R\partial Q}\right]dR\,.dQ$$

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$-n_2 ab \iint \left(q\beta_1 h + \frac{N_x}{2\alpha^2}{\beta_1}^2 \left(\frac{dh}{dR}\right)^2 + {\beta_1}^2 \frac{m\lambda^2}{2} h^2 \right) dR\, dQ \qquad (29b)$$

$$\Pi_3 = \frac{1.5ab}{2a^4} \iint \left[A_{23} \frac{\beta_2\beta_3}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} + A_{23} \frac{{\beta_3}^2}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} - 3B_{23} \frac{\beta_1\beta_3}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} - B_{23} \frac{\beta_1\beta_2}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} + 2D_{23} \frac{{\beta_1}^2}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} \right] dR\,.dQ - n_3 ab \iint \left(q\beta_1 h + \frac{N_x}{2a^2}{\beta_1}^2 \left(\frac{dh}{dR}\right)^2 + {\beta_1}^2 \frac{m\lambda^2}{2} h^2 \right) dR\, dQ \quad (29c)$$

Minimizing equations 29a, 29b and 29c with respect to β1 and rearrange gives respectively:

$$\frac{d\Pi_1}{d\beta_1} = 0 = \frac{ab}{2a^4} \iint \left[-2\left(B_{11}\beta_2 \left(\frac{\partial^2 h}{\partial R^2}\right)^2 + (B_{12}+2B_{33}) \frac{\beta_2}{\alpha^2} \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + (B_{12}+2B_{33}) \frac{\beta_3}{\alpha^2} \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + B_{22} \frac{\beta_3}{\alpha^4} \left(\frac{\partial^2 h}{\partial Q^2}\right)^2 \right) + 2\beta_1 \left(D_{11} \left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{\alpha^2}(D_{12}+2D_{33}) \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{D_{22}}{\alpha^4} \left(\frac{\partial^2 h}{\partial Q^2}\right)^2 \right) \right] dR dQ$$

$$-n_1 ab \iint \left(qh + \frac{N_x}{\alpha^2}\beta_1 \left(\frac{dh}{dR}\right)^2 + \beta_1 m\lambda^2 h^2 \right) dR\, dQ \qquad (29a)$$

$$\frac{d\Pi_1}{d\beta_1} = 0 = -\left(B_{11}\beta_2 k_x + (B_{12}+2B_{33}) \frac{\beta_2}{\alpha^2} k_{xy} + (B_{12}+2B_{33}) \frac{\beta_3}{\alpha^2} k_{xy} + B_{22} \frac{\beta_3}{\alpha^4} k_y \right) + \beta_1 \left(D_{11} k_x + \frac{2}{\alpha^2}(D_{12}+2D_{33}) k_{xy} + \frac{D_{22}}{\alpha^4} k_y \right) - n_1 a^4 \left(q k_q + \frac{N_x}{a^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda \right) \quad (30a)$$

$$\frac{d\Pi_2}{d\beta_1} = 0 = \frac{1.5ab}{2a^4} \iint \left[-3B_{13} \frac{\beta_2}{\alpha} \frac{\partial^2 h}{\partial R^2} \frac{\partial^2 h}{\partial R \partial Q} - B_{13} \frac{\beta_3}{\alpha} \cdot \frac{\partial^2 h}{\partial R^2} \frac{\partial^2 h}{\partial R \partial Q} + 4D_{13} \frac{\beta_1}{\alpha} \frac{\partial^2 h}{\partial R^2} \frac{\partial^2 h}{\partial R \partial Q} \right] dR\,.dQ - n_2 ab \iint \left(qh + \frac{N_x}{a^2}\beta_1 \left(\frac{dh}{dR}\right)^2 + \beta_1 m\lambda^2 h^2 \right) dR\, dQ \qquad (30b)$$

$$\frac{d\Pi_2}{d\beta_1} = 0 = \left(3D_{13} \frac{\beta_1}{\alpha} - 2.25B_{13} \frac{\beta_2}{\alpha} - 0.75B_{13} \frac{\beta_3}{\alpha} \right) k_{xxy} - n_2 a^4 \left(q k_x + \frac{N_x}{a^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda \right) \qquad (30b)$$

$$\frac{d\Pi_3}{d\beta_1} = 0 = \frac{1.5ab}{2a^4} \iint \left[-3B_{23} \frac{\beta_3}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} - B_{23} \frac{\beta_2}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} + 4D_{23} \frac{\beta_1}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} \right] dR\,.dQ - n_3 ab \iint \left(qh + \frac{N_x}{a^2}\beta_1 \left(\frac{dh}{dR}\right)^2 + \beta_1 m\lambda^2 h^2 \right) dR\, dQ \quad (30c)$$

$$\frac{d\Pi_3}{d\beta_1} = 0 = \left(3D_{23} \frac{\beta_1}{\alpha^3} - 0.75B_{23} \frac{\beta_2}{\alpha^3} - 2.25B_{23} \frac{\beta_3}{\alpha^3} \right) k_{xyy} - n_3 a^4 \left(q k_x + \frac{N_x}{a^2}\beta_1 k_N + \beta_1 m\lambda^2 k_\lambda \right) \quad (30c)$$

Minimizing equations 29a, 29b and 29c with respect to β_2 gives:

$$\frac{d\Pi_1}{d\beta_2} = \frac{ab}{2a^4} \iint \left[\left(2A_{11}\beta_2 \left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{\alpha^2}[A_{12}\beta_3 + A_{33}\beta_3 + A_{33}\beta_2] \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 \right) - 2\left(B_{11}\beta_1 \left(\frac{\partial^2 h}{\partial R^2}\right)^2 + (B_{12}+2B_{33}) \frac{\beta_1}{\alpha^2} \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 \right) \right] dR dQ = 0 \qquad (31a)$$

$$\frac{d\Pi_1}{d\beta_2} = \frac{ab}{a^4} \left[\left(A_{11}\beta_2 k_x + \frac{1}{\alpha^2}[A_{12}\beta_3 + A_{33}\beta_3 + A_{33}\beta_2] k_{xy} \right) - \left(B_{11}\beta_1 k_x + (B_{12}+2B_{33}) \frac{\beta_1}{\alpha^2} k_{xy} \right) \right] = 0 \qquad (31a)$$

$$\frac{d\Pi_2}{d\beta_2} = \frac{1.5ab}{2a^4} \iint \left[A_{13} \frac{{\beta_2}^2}{\alpha} \frac{\partial^2 h}{\partial R^2} \cdot \frac{\partial^2 h}{\partial R \partial Q} + A_{13} \frac{\beta_2\beta_3}{\alpha} \frac{\partial^2 h}{\partial R^2} \cdot \frac{\partial^2 h}{\partial R \partial Q} - 3B_{13} \frac{\beta_1\beta_2}{\alpha} \frac{\partial^2 h}{\partial R^2} \frac{\partial^2 h}{\partial R \partial Q} \right] dR\,.dQ = 0 \qquad (31b)$$

$$\frac{d\Pi_2}{d\beta_2} = \frac{ab}{a^4} \left[1.5A_{13} \frac{\beta_2}{\alpha} k_{xxy} + 0.75A_{13} \frac{\beta_3}{\alpha} k_{xxy} - 2.25B_{13} \frac{\beta_1}{\alpha} k_{xxy} \right] = 0 \qquad (31b)$$

$$\frac{d\Pi_3}{d\beta_2} = 0 = \frac{1.5ab}{2a^4} \iint \left[A_{23} \frac{\beta_3}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} - B_{23} \frac{\beta_1}{\alpha^3} \frac{\partial^2 h}{\partial R \partial Q} \cdot \frac{\partial^2 h}{\partial Q^2} \right] dR\,.dQ \qquad (31c)$$

$$\frac{d\Pi_3}{d\beta_2} = 0 = \frac{ab}{a^4} \left[0.75A_{23} \frac{\beta_3}{\alpha^3} k_{xyy} - 0.75B_{23} \frac{\beta_1}{\alpha^3} k_{xyy} \right] \qquad (31c)$$

Minimizing equations 29a, 29b and 29c with respect to β_3 gives:

International Journal of Scientific & Engineering Research, Volume 8, Issue 3, March-2020
ISSN 2229-5518

$$\frac{\mathrm{d}\Pi_1}{\mathrm{d}\beta_3} = \frac{ab}{\alpha^4}\iint\left[\left(\frac{1}{\alpha^2}[A_{12}\beta_2 + A_{33}\beta_2 + A_{33}\beta_3]\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 + A_{22}\frac{\beta_3}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right) - \left((B_{12} + 2B_{33})\frac{\beta_1}{\alpha^2}\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2 + B_{22}\frac{\beta_1}{\alpha^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\right)\right] dRdQ \; = 0 \qquad (32a)$$

$$\frac{\mathrm{d}\Pi_1}{\mathrm{d}\beta_3} = \frac{ab}{\alpha^4}\left[A_{22}\frac{\beta_3}{\alpha^4}k_y + \frac{1}{\alpha^2}[A_{12}\beta_2 + A_{33}\beta_2 + A_{33}\beta_3]k_{xy} - B_{22}\frac{\beta_1}{\alpha^4}k_y - (B_{12} + 2B_{33})\frac{\beta_1}{\alpha^2}k_{xy}\right] = 0 \qquad (32a)$$

$$\frac{\mathrm{d}\Pi_2}{\mathrm{d}\beta_3} = \frac{1.5ab}{2a^4}\iint\left[A_{13}\frac{\beta_2\beta_3}{\alpha}\frac{\partial^2 h}{\partial R^2}.\frac{\partial^2 h}{\partial R\partial Q} - B_{13}\frac{\beta_1\beta_3}{\alpha}.\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q}\right]\; dR\,.\,dQ \qquad (32b)$$

$$\frac{\mathrm{d}\Pi_2}{\mathrm{d}\beta_3} = 0 = \frac{ab}{\alpha^4}\left[0.75A_{13}\frac{\beta_2}{\alpha}k_{xxy} - 0.75B_{13}\frac{\beta_1}{\alpha}.k_{xxy}\right] \qquad (32b)$$

$$\frac{\mathrm{d}\Pi_3}{\mathrm{d}\beta_3} = 0 = \frac{ab}{\alpha^4}\iint\left[0.75A_{23}\frac{\beta_2}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} + 1.5A_{23}\frac{\beta_3}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2} - 2.25B_{23}\frac{\beta_3}{\alpha^3}\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2}\right]\; dR\,.\,dQ \qquad (32c)$$

$$\frac{\mathrm{d}\Pi_3}{\mathrm{d}\beta_3} = 0 = \frac{ab}{\alpha^4}\left[0.75A_{23}\frac{\beta_2}{\alpha^3}k_{xyy} + 1.5A_{23}\frac{\beta_3}{\alpha^3}k_{xyy} - 2.25B_{23}\frac{\beta_3}{\alpha^3}k_{xyy}\right] \qquad (32c)$$

$$Where: k_x = \iint\left(\frac{\partial^2 h}{\partial R^2}\right)^2\; dR\,.\,dQ \;:\; k_{xy} = \iint\left(\frac{\partial^2 h}{\partial R\partial Q}\right)^2\; dR\,.\,dQ \;:\; k_y = \iint\left(\frac{\partial^2 h}{\partial Q^2}\right)^2\; dR\,.\,dQ$$

$$k_{xxy} = \iint\frac{\partial^2 h}{\partial R^2}\frac{\partial^2 h}{\partial R\partial Q}\; dR\,.\,dQ : k_{xyy} = \iint\frac{\partial^2 h}{\partial R\partial Q}.\frac{\partial^2 h}{\partial Q^2}\; dR\,.\,dQ \;:\; k_q = \iint h\;\; dR\,.\,dQ$$

$$k_N = \iint\left(\frac{dh}{dR}\right)^2\; dR\,.\,dQ : \; k_\lambda = \iint h^2\;\; dR\,.\,dQ$$